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On multiple $\mathbb{Z}_p$-extensions of imaginary abelian quartic fields

Tsuyoshi Itoh [1]

College of Science and Engineering, Ritsumeikan University, 1-1-1 Noji Higashi, Kusatsu, Shiga 525-8577, Japan

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ABSTRACT

Let k be an imaginary abelian quartic field and p an odd prime which splits completely in k. We give a sufficient condition for the validity of Greenberg's Generalized Conjecture (for multiple $\mathbb{Z}_p$-extensions) for k and p.

1. Introduction

First, we shall briefly explain about Iwasawa theory for multiple $\mathbb{Z}_p$-extensions (for details, see [5]). Let k be an algebraic number field and p a prime number. Let K/k be a $\mathbb{Z}_p^d$-extension (where d is a positive integer). We denote by $L(K)$ the maximal unramified abelian pro-p extension of K. Then it is known that the Galois group $\mathrm{Gal}(L(K)/K)$ is a finitely generated torsion module over the complete group ring $\mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$. Since $\mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$ is isomorphic to the d-variable power series ring $\Lambda_d := \mathbb{Z}_p[[T_1, T_2, \dots, T_d]]$, we may regard an element of $\mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$ as in Λ_d (under a fixed isomorphism).

A finitely generated $\mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$-module M is called pseudo-null if there are nontrivial two annihilators $f, g \in \mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$ of M such that f and g are relatively prime (cf. [8]). For a finitely generated torsion $\mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$-module M, it is known that M is pseudo-null if and only if

E-mail address: tsuyoshi.itoh@it-chiba.ac.jp.

[1] Current address: Division of Mathematics, Education Center, Faculty of Social Systems Science, Chiba Institute of Technology, 2-1-1 Shibazono, Narashino, Chiba 275-0023, Japan.

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$\mathrm{Ext}^1_{\mathbb{Z}_p[\![\mathrm{Gal}(K/k)]\!]}(M, \mathbb{Z}_p[\![\mathrm{Gal}(K/k)]\!])$ is trivial. (This fact follows from, e.g., [20, p. 16, Proposition 8].) When $d = 1$, M is pseudo-null if and only if the order of M is finite (see, e.g., [24, Lemma 15.17]). Let $\widetilde{k}$ be the composite of all $\mathbb{Z}_p$-extensions of k. It is known that $\widetilde{k}/k$ is a $\mathbb{Z}_p^d$-extension with some positive integer d (see, e.g., [24, Theorem 13.4]). In this case, the following conjecture is known. It is often called Greenberg's Generalized Conjecture (GGC).

Conjecture. *(See Greenberg [8].) The Galois group* $\mathrm{Gal}(L(\widetilde{k})/\widetilde{k})$ *is pseudo-null as a module over* $\mathbb{Z}_p[\![\mathrm{Gal}(\widetilde{k}/k)]\!]$ *for every algebraic number field k and every prime number p.*

If k is totally real and Leopoldt's conjecture holds for k and p, GGC is equivalent to (usual) Greenberg's Conjecture (see [6,8]). We shall state some of known results for the case that k is not totally real. When k is an imaginary quadratic field, Minardi [16,17] showed that if p does not divide the class number of k then GGC holds for k and p. The author [10] gave some families of imaginary quadratic fields which satisfy GGC for $p = 2$. When k is an imaginary biquadratic field and p is an odd prime number which splits at most two distinct primes in k, Bandini [1] gave some sufficient conditions for which GGC holds. See also [2]. When k is the pth cyclotomic field, it is known that GGC holds for k and p if $p < 1000$ (see McCallum and Sharifi [15], Sharifi [21,22]).

In the present paper, we shall give a sufficient condition for the case that the base field k is an imaginary abelian quartic field (namely, an imaginary cyclic quartic field or an imaginary biquadratic field) and p is an odd prime number which splits completely in k. The following is our main theorem.

Theorem 1.1. *Let k be an imaginary abelian quartic filed and p an odd prime number which splits completely in k. Assume that the class number of k is not divisible by p. We denote by* k^+ *the maximal real subfield of k and by* k^+_∞ *the cyclotomic* $\mathbb{Z}_p$*-extension of k. If all of the Iwasawa invariants* λ, μ, *and* ν *of* k^+_∞/k^+ *are zero (that is,* $\mathrm{Gal}(L(k^+_\infty)/k^+_\infty)$ *is trivial), then GGC holds for k and p.*

Here we shall give an example. Let k be the fifth cyclotomic field, and p a prime number which splits completely in k (that is, $p \equiv 1 \pmod 5$). Note that the class number of k is 1, and the maximal real subfield of k is $\mathbb{Q}(\sqrt{5})$. Assume that all of the Iwasawa invariants λ, μ, and ν of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}(\sqrt{5})$ are zero. Then GGC for k and p holds by Theorem 1.1. This assumption is satisfied for many primes p. (See, e.g., Sumida-Takahashi [23]. Many real quadratic fields and primes satisfying $\lambda = \mu = \nu = 0$ are also given there.)

The outline of the proof is as follows. First, we take a certain $\mathbb{Z}_p$-extension $K^{(1)}/k$ and a certain $\mathbb{Z}_p^2$-extension $K^{(2)}/k$. We show that $\mathrm{Gal}(L(K^{(1)})/K^{(1)})$ is trivial (Corollary 3.3). From this, we can see that $\mathrm{Gal}(L(K^{(2)})/K^{(2)})$ is pseudo-null as a $\mathbb{Z}_p[\![\mathrm{Gal}(K^{(2)}/k)]\!]$-module (Proposition 4.1). Finally, we will prove Theorem 1.1 by using these results. Our method can be considered as a generalization of the proof of Minardi's result which is mentioned above (see Sections 4 and 5).

2. Preliminaries

Let p be an odd prime number and F an algebraic number field. We denote by $E(F)$ the group of units of F, and by $A(F)$ the Sylow p-subgroup of the ideal class group of F. For a prime $\mathfrak{p}$ of F, let $U_\mathfrak{p}$ (resp. $U^1_\mathfrak{p}$) be the group of units (resp. principal units) in the completion of F at $\mathfrak{p}$. Let F' be an algebraic extension of the field $\mathbb{Q}$ of rational numbers. We write $L(F')$ for the maximal unramified abelian pro-p extension of F', and $\mathcal{M}_\mathfrak{p}(F')$ for the maximal pro-p abelian extension of F' which is unramified outside the primes lying above $\mathfrak{p}$ (where F is a finite extension of $\mathbb{Q}$ contained in F', and $\mathfrak{p}$ is a prime of F). If F'/F is a $\mathbb{Z}_p$-extension, we denote by $\lambda(F'/F)$, $\mu(F'/F)$, and $\nu(F'/F)$ the Iwasawa λ-invariant, μ-invariant, and ν-invariant of F'/F, respectively. Let T be a non-empty finite set of primes of F lying above p (it is not necessary that T contains all primes lying above p). We put $U_T = \prod_{\mathfrak{p}\in T} U_\mathfrak{p}$ and $U^1_T = \prod_{\mathfrak{p}\in T} U^1_\mathfrak{p}$. We define the map $\varphi : E(F) \to U_T$ by embedding diagonally. We denote by $\mathcal{E}(F)_T$ the closure of $\varphi(E(F)) \cap U^1_T$ in U^1_T. For the case that $T = \{\mathfrak{p}\}$, we will often write $\mathcal{E}(F)_\mathfrak{p}$ instead of $\mathcal{E}(F)_T$.

The following result will be used frequently in our proof of the main theorem.

Proposition 2.1. *(See Maire [14].) Let k be an imaginary abelian quartic field and p an odd prime number which splits completely in k. We take a prime $\mathfrak{P}$ of k lying above p. Let K be a finite abelian extension over k. Then $\mathcal{M}_{\mathfrak{P}}(K)/K$ is a finite extension.*

Proof. This is a special case of [14, Theorem 25]. $\square$

Lemma 2.2. *Let k be an imaginary abelian quartic field and p an odd prime number which splits completely in k. For distinct primes $\mathfrak{P}$ and $\mathfrak{P}'$ of k lying above p, there exists a unique $\mathbb{Z}_p$-extension K/k which is unramified outside $\{\mathfrak{P}, \mathfrak{P}'\}$. Moreover, every prime of k lying above p is finitely decomposed in K.*

Proof. We put $T = \{\mathfrak{P}, \mathfrak{P}'\}$. Let $\mathcal{M}_T(k)$ be the maximal abelian pro-p extension over k which is unramified outside the primes contained in T. Then by class field theory, we have the following exact sequence (see [9, Proposition 3.1], [14, Theorem 5]):

$$0 \to U_T^1/\mathcal{E}(k)_T \to \mathrm{Gal}\big(\mathcal{M}_T(k)/k\big) \to A(k) \to 0.$$

In this case, the $\mathbb{Z}_p$-rank of U_T^1 is 2. Since the $\mathbb{Z}$-rank of $E(k)$ is 1, the $\mathbb{Z}_p$-rank of $\mathcal{E}(k)_T$ is also 1. We see that the $\mathbb{Z}_p$-rank of $\mathrm{Gal}(\mathcal{M}_T(k)/k)$ is 1 because $A(k)$ is finite. The existence and the uniqueness have been proved.

We shall show the latter part. We note that both of $\mathfrak{P}$ and $\mathfrak{P}'$ must be ramified infinitely by Proposition 2.1. Hence both of $\mathfrak{P}$ and $\mathfrak{P}'$ are finitely decomposed. Next, we take an arbitrary prime $\mathfrak{P}''$ of k which is lying above p but different from $\mathfrak{P}$ and $\mathfrak{P}'$. Let $\mathcal{M}'$ be the maximal abelian pro-p extension over k which is unramified outside the primes contained in T and $\mathfrak{P}''$ splits completely. It is sufficient to show that the $\mathbb{Z}_p$-rank of $\mathrm{Gal}(\mathcal{M}'/k)$ is zero. Let $E(k)''$ be the group of $\mathfrak{P}''$-units in k and $\varphi(E(k)'')$ the diagonal image of $E(k)''$ in U_T. We denote by $\mathcal{E}(k)''_T$ the closure of $\varphi(E(k)'') \cap U_T^1$ in U_T^1. By class field theory, the $\mathbb{Z}_p$-rank of $\mathrm{Gal}(\mathcal{M}'/k)$ is equal to the $\mathbb{Z}_p$-rank of $U_T^1/\mathcal{E}(k)''_T$ (see [14, Theorem 5]). Hence, we shall show that the $\mathbb{Z}_p$-rank of $\mathcal{E}(k)''_T$ is exactly 2. Assume that the $\mathbb{Z}_p$-rank of $\mathcal{E}(k)''_T$ is 1. Then there exists an element π of $E(k)''$ which satisfies the following properties:

- $(\mathfrak{P}'')^h = (\pi)$ with a positive integer h, and
- for every positive integer n,

$$\pi \equiv \varepsilon_n \pmod{\mathfrak{P}^n}, \qquad \pi \equiv \varepsilon_n \pmod{\mathfrak{P}'^n}$$

  with a unit ε_n of k.

It is known that the index $(E(k) : W(k)E(k^+))$ is 1 or 2 (see, e.g. [24, Theorem 4.12]), where $W(k)$ is the group of roots of unity in k. Hence we see that $\varepsilon_n^{4w} \in k^+$ and $N_{k^+/\mathbb{Q}}(\varepsilon_n^{4w}) = 1$ for all n, where w is the order of $W(k)$. We take an element σ of $\mathrm{Gal}(k/\mathbb{Q})$ which satisfies $(\mathfrak{P}')^\sigma = \mathfrak{P}$. If σ is contained in $\mathrm{Gal}(k/k^+)$, then $\pi^{4w} \equiv (\pi^{4w})^\sigma \pmod{\mathfrak{P}^n}$ for all n because $\varepsilon_n^{4w} = (\varepsilon_n^{4w})^\sigma$. Hence $\pi^{4w} = (\pi^{4w})^\sigma$. It is a contradiction because $(\pi^\sigma) = ((\mathfrak{P}'')^\sigma)^h$ and $(\mathfrak{P}'')^\sigma \neq \mathfrak{P}''$. Otherwise, we can see that $\pi^{4w}(\pi^{4w})^\sigma \equiv 1 \pmod{\mathfrak{P}^n}$ for all n because $\varepsilon_n^{4w}(\varepsilon_n^{4w})^\sigma = 1$. Hence $\pi^{4w}(\pi^{4w})^\sigma = 1$. It is also a contradiction since π generates a power of a prime lying above p. This shows that the $\mathbb{Z}_p$-rank of $\mathcal{E}(k)_T$ is greater than 1, and then $\mathrm{Gal}(\mathcal{M}'/k)$ is a finite group. Hence $\mathfrak{P}''$ cannot be infinitely decomposed. We have shown the lemma completely. $\square$

From now on, we fix the following notation:

k: an imaginary abelian quartic number field;
k^+: the maximal real subfield of k;

p: an odd prime number which splits completely in k;
$\mathfrak{p}_1$, $\mathfrak{p}_2$: distinct primes of k^+ lying above p;
$\mathfrak{P}_i$, $\mathfrak{P}_i'$: distinct primes of k lying above $\mathfrak{p}_i$ $(i = 1, 2)$.

By Lemma 2.2, there exists a unique $\mathbb{Z}_p$-extension $K^{(1)}/k$ which is unramified outside $\{\mathfrak{P}_1, \mathfrak{P}_2\}$. We can also show that there exists a unique $\mathbb{Z}_p^2$-extension $K^{(2)}/k$ which is unramified outside $\{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_1'\}$ by using the same method given in the proof of Lemma 2.2. Let $\widetilde{k}$ be the composite of all $\mathbb{Z}_p$-extensions of k. It is known that $\widetilde{k}/k$ is a $\mathbb{Z}_p^3$-extension (see, e.g., [24, Theorem 13.4]). Clearly, we can see that $K^{(1)} \subset K^{(2)} \subset \widetilde{k}$.

The following result is also important for the proof of our main theorem. This is a special case of [20, p. 12, Lemme 4].

Proposition 2.3. *(See Perrin-Riou [20].) Let d be an integer which satisfies $d \geqslant 2$, K/k a $\mathbb{Z}_p^d$-extension, and K'/k a $\mathbb{Z}_p^{d-1}$-extension which satisfies $K' \subset K$. Let M be a finitely generated torsion $\mathbb{Z}_p[\![\mathrm{Gal}(K/k)]\!]$-module and $M_{\mathrm{Gal}(K/K')}$ the $\mathrm{Gal}(K/K')$-coinvariant quotient module of M. If $M_{\mathrm{Gal}(K/K')}$ is pseudo-null as a module over $\mathbb{Z}_p[\![\mathrm{Gal}(K'/k)]\!]$, then M is pseudo-null as a module over $\mathbb{Z}_p[\![\mathrm{Gal}(K/k)]\!]$.*

3. The $\mathbb{Z}_p$-extension $K^{(1)}/k$

At first, we shall state the following known result.

Theorem 3.1. *(See Fukuda and Komatsu [3], Hachimori [9].) Let F be a real quadratic field, p an odd prime number which splits in F, and F_∞ the cyclotomic $\mathbb{Z}_p$-extension of F. We denote by ε_F the fundamental unit of F and we regard ε_F as an element of $\mathbb{Z}_p$ (with a fixed embedding of k in the field $\mathbb{Q}_p$ of p-adic numbers). Then, $\lambda(F_\infty/F) = \mu(F_\infty/F) = \nu(F_\infty/F) = 0$ if and only if p does not divide the class number of F and $\varepsilon_F^{p-1} \not\equiv 1 \pmod{p^2\mathbb{Z}_p}$.*

We shall show a non-cyclotomic analog of a part of the above theorem.

Proposition 3.2. *If the class number of k is not divisible by p, $U^1_{\mathfrak{P}_1} = \mathcal{E}(k)_{\mathfrak{P}_1}$, and $U^1_{\mathfrak{P}_2} = \mathcal{E}(k)_{\mathfrak{P}_2}$, then $\lambda(K^{(1)}/k) = \mu(K^{(1)}/k) = \nu(K^{(1)}/k) = 0$.*

Proof. We have the following exact sequence:

$$0 \to U^1_{\mathfrak{P}_1}/\mathcal{E}(k)_{\mathfrak{P}_1} \to \mathrm{Gal}\big(\mathcal{M}_{\mathfrak{P}_1}(k)/k\big) \to A(k) \to 0$$

by class field theory (recall also the first part of the proof of Lemma 2.2). From the assumptions, we see that $\mathrm{Gal}(\mathcal{M}_{\mathfrak{P}_1}(k)/k)$ is trivial. Hence there is no abelian p-extension which is unramified outside $\mathfrak{P}_1$. This implies that $\mathfrak{P}_2$ is totally ramified in $K^{(1)}/k$. Similarly, we can see that $\mathfrak{P}_1$ is also totally ramified in $K^{(1)}/k$.

The rest of the proof can be shown by using the idea given in [9] (see also Chapter 13 of [24]). It is sufficient to show that $L(K^{(1)})$ coincides with $K^{(1)}$. We note that $\mathrm{Gal}(\mathcal{M}_{\mathfrak{P}_1}(K^{(1)})/K^{(1)})$ is a compact $\mathbb{Z}_p[\![\mathrm{Gal}(K^{(1)}/k)]\!]$-module. Let $I_{\mathfrak{P}_2}$ be the inertia group of $\mathrm{Gal}(\mathcal{M}_{\mathfrak{P}_1}(K^{(1)})/k)$ for some prime lying above $\mathfrak{P}_2$. We note that $I_{\mathfrak{P}_2} \cap \mathrm{Gal}(\mathcal{M}_{\mathfrak{P}_1}(K^{(1)})/K^{(1)})$ is trivial because $\mathcal{M}_{\mathfrak{P}_1}(K^{(1)})/K^{(1)}$ is unramified outside the primes lying above $\mathfrak{P}_1$. Since $\mathfrak{P}_2$ is totally ramified in $K^{(1)}/k$, we see that $K^{(1)} \cap \mathcal{M}_{\mathfrak{P}_1}(k) = k$. From these facts, we can see that the $\mathrm{Gal}(K^{(1)}/k)$-coinvariant quotient of $\mathrm{Gal}(\mathcal{M}_{\mathfrak{P}_1}(K^{(1)})/K^{(1)})$ is isomorphic to $\mathrm{Gal}(\mathcal{M}_{\mathfrak{P}_1}(k)/k)$. Since $\mathrm{Gal}(\mathcal{M}_{\mathfrak{P}_1}(k)/k)$ is trivial, $\mathrm{Gal}(\mathcal{M}_{\mathfrak{P}_1}(K^{(1)})/K^{(1)})$ is also trivial by Nakayama's lemma. We have shown the proposition because $L(K^{(1)})$ is contained in $\mathcal{M}_{\mathfrak{P}_1}(K^{(1)})$. $\square$

Remark. For the case that $p=2$ and k is biquadratic, a similar result is given in [12]. Goto [4] independently proved a similar result for more general situations.

Let k_∞^+ be the cyclotomic $\mathbb{Z}_p$-extension of the real quadratic filed k^+. As a consequence of the above results, we have the following:

Corollary 3.3. *We assume that the class number of k is not divisible by p and $\lambda(k_\infty^+/k^+)=\mu(k_\infty^+/k^+)=\nu(k_\infty^+/k^+)=0$. Then $\lambda(K^{(1)}/k)=\mu(K^{(1)}/k)=\nu(K^{(1)}/k)=0$. (That is, $\mathrm{Gal}(L(K^{(1)})/K^{(1)})$ is trivial.)*

Proof. Since p splits completely in k, we see that $U^1_{\mathfrak{P}_1}$ is isomorphic to $U^1_{\mathfrak{p}_1}$. By the assumption and Theorem 3.1, we obtain that $U^1_{\mathfrak{p}_1}=\mathcal{E}(k^+)_{\mathfrak{p}_1}$. Hence we see that $U^1_{\mathfrak{P}_1}=\mathcal{E}(k)_{\mathfrak{P}_1}$. By using the same method, we also see that $U^1_{\mathfrak{P}_2}=\mathcal{E}(k)_{\mathfrak{P}_2}$. The assertion follows from Proposition 3.2. $\square$

4. The $\mathbb{Z}_p^2$-extension $K^{(2)}/k$

Recall that $K^{(2)}$ is the unique $\mathbb{Z}_p^2$-extension of k which is unramified outside $\{\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_1'\}$. We put $X_1=\mathrm{Gal}(L(K^{(1)})/K^{(1)})$, $X_2=\mathrm{Gal}(L(K^{(2)})/K^{(2)})$, and $G=\mathrm{Gal}(K^{(2)}/K^{(1)})\cong\mathbb{Z}_p$.

Proposition 4.1. *Assume that neither $\mathfrak{P}_1$ nor $\mathfrak{P}_2$ splits in $K^{(1)}/k$. If the order of X_1 is finite, then X_2 is pseudo-null $\mathbb{Z}_p[\![\mathrm{Gal}(K^{(2)}/k)]\!]$-module.*

Proof. Our proof is quite similar to the proof of Minardi's result [16,17] (for the proof, it is convenient to see [19, §4]). However, we shall give a detailed proof because there are some points which must be confirmed carefully.

We denote by $(X_2)_G$ the G-coinvariant quotient module of X_2. By Proposition 2.3, it is sufficient to show that the order of $(X_2)_G$ is finite. Let L_1 be the maximal subfield of $L(K^{(2)})$ which is abelian over $K^{(1)}$. Then $\mathrm{Gal}(L_1/K^{(2)})$ is isomorphic to $(X_2)_G$.

By Lemma 2.2, the prime $\mathfrak{P}_1'$ of k is finitely decomposed in $K^{(1)}$. Then we take a positive integer n_0 such that the decomposition field of $K^{(1)}/k$ for $\mathfrak{P}_1'$ coincides with $K^{(1)}_{n_0}$ (where $K^{(1)}_{n_0}$ is the n_0th layer of $K^{(1)}/k$).

We put $\mathcal{I}_1=\sum_P I_P$, where P runs all primes of $K^{(1)}$ which ramify in $L_1/K^{(1)}$, and I_P denotes the inertia group of $\mathrm{Gal}(L_1/K^{(1)})$ for P. We have the following exact sequence of $\mathbb{Z}_p[\![\mathrm{Gal}(K^{(1)}/k)]\!]$-modules:

$$0\to\mathcal{I}_1\to\mathrm{Gal}\big(L_1/K^{(1)}\big)\to\mathrm{Gal}\big(L\big(K^{(1)}\big)/K^{(1)}\big)\to 0.$$

We note that all primes of $K^{(1)}$ which ramify in $L_1/K^{(1)}$ are the primes lying above $\mathfrak{P}_1'$. We also note that $\mathfrak{P}_1'$ is finitely decomposed in $K^{(1)}$ by Lemma 2.2. Hence $\mathcal{I}_1$ is finitely generated over $\mathbb{Z}_p$ because each I_P is isomorphic to the additive group of $\mathbb{Z}_p$. This implies that $\mathcal{I}_1$ is a finitely generated torsion $\mathbb{Z}_p[\![\mathrm{Gal}(K^{(1)}/k)]\!]$-module. By the assumption that $X_1=\mathrm{Gal}(L(K^{(1)})/K^{(1)})$ is finite, we also see that $\mathrm{Gal}(L_1/K^{(1)})$ is a finitely generated torsion $\mathbb{Z}_p[\![\mathrm{Gal}(K^{(1)}/k)]\!]$-module, and $\mathcal{I}_1$ is pseudo-isomorphic to $\mathrm{Gal}(L_1/K^{(1)})$. Let L_1' be the subfield of L_1 corresponding to the maximal pseudo-null (that is, finite) submodule of $\mathrm{Gal}(L_1/K^{(1)})$.

We claim that L_1' is an abelian extension over $K^{(1)}_{n_0}$. Take an arbitrary prime P_1 of $K^{(1)}$ lying above $\mathfrak{P}_1'$, and let I_{P_1} be the inertia group of $\mathrm{Gal}(L_1'/K^{(1)})$ for P_1. Then $\mathrm{Gal}(K^{(1)}/K^{(1)}_{n_0})$ acts on I_{P_1}. Since $L_1/K^{(2)}$ is an unramified extension, the restriction map $I_{P_1}\to\mathrm{Gal}(K^{(2)}/K^{(1)})$ is injective. We see that $\mathrm{Gal}(K^{(1)}/K^{(1)}_{n_0})$ acts trivially on I_{P_1} for each prime P_1 lying above $\mathfrak{P}_1'$. Then, $\mathrm{Gal}(K^{(1)}/K^{(1)}_{n_0})$ acts trivially on $\mathcal{I}_1$. Since pseudo-isomorphism satisfies the symmetric law for torsion $\mathbb{Z}_p[\![\mathrm{Gal}(K^{(1)}/k)]\!]$-modules (see [18,24]) and $\mathrm{Gal}(L_1'/K^{(1)})$ has no nontrivial pseudo-null submodule, there is an injection (as $\mathbb{Z}_p[[\mathrm{Gal}(K^{(1)}/k)]]$-modules) from $\mathrm{Gal}(L_1'/K^{(1)})$ to $\mathcal{I}_1$. Hence $\mathrm{Gal}(K^{(1)}/K_{n_0}^{(1)})$ also acts trivially on $\mathrm{Gal}(L_1'/K^{(1)})$. The claim follows.

By the assumption, neither $\mathfrak{P}_1$ nor $\mathfrak{P}_2$ splits in $K^{(1)}/k$. We denote by $I_{\mathfrak{P}_1}$ (resp. $I_{\mathfrak{P}_2}$) the inertia group of $L_1'/K_{n_0}^{(1)}$ for the prime of $K_{n_0}^{(1)}$ lying above $\mathfrak{P}_1$ (resp. $\mathfrak{P}_2$). Note that the $\mathbb{Z}_p$-rank of $I_{\mathfrak{P}_1}$ (resp. $I_{\mathfrak{P}_2}$) is 1 because the prime of $K^{(1)}$ lying above $\mathfrak{P}_1$ (resp. $\mathfrak{P}_2$) is unramified in L_1'. Let M_1 be the fixed field of L_1' by $\sum_{i=1,2} I_{\mathfrak{P}_i}$. Then we have the following exact sequence:

$$0 \to \sum_{i=1,2} I_{\mathfrak{P}_i} \to \mathrm{Gal}\bigl(L_1'/K_{n_0}^{(1)}\bigr) \to \mathrm{Gal}\bigl(M_1/K_{n_0}^{(1)}\bigr) \to 0.$$

Since M_1 is an abelian extension of $K_{n_0}^{(1)}$ which is unramified outside the primes lying above $\mathfrak{P}_1'$, we see that the order of $\mathrm{Gal}(M_1/K_{n_0}^{(1)})$ is finite by Proposition 2.1. Hence, we can conclude that the $\mathbb{Z}_p$-rank of $\mathrm{Gal}(L_1'/K_{n_0}^{(1)})$ is exactly 2. This implies that L_1' is a finite extension over $K^{(2)}$ because the $\mathbb{Z}_p$-rank of $\mathrm{Gal}(K^{(2)}/K_{n_0}^{(1)})$ is 2. Since L_1/L_1' is a finite extension, L_1' is also a finite extension over $K^{(2)}$. Consequently, the order of $(X_2)_G$ is finite, and then the proposition follows. □

5. Proof of Theorem 1.1

Let the notation be as in the previous sections. Recall that k is an imaginary abelian quartic field and p is an odd prime which splits completely in k. We assume that the class number of k is not divisible by p, and $\lambda(k_\infty^+/k^+) = \mu(k_\infty^+/k^+) = \nu(k_\infty^+/k^+) = 0$. By the results given in Section 3, we see that $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are totally ramified in $K^{(1)}/k$ and X_1 is trivial. Hence X_2 is pseudo-null as a $\mathbb{Z}_p[[\mathrm{Gal}(K^{(2)}/k)]]$-module by Proposition 4.1.

The outline of the following argument is the same as the proof of Proposition 4.1. We put $\widetilde{X} = \mathrm{Gal}(L(\widetilde{k})/\widetilde{k})$ and $G_2 = \mathrm{Gal}(\widetilde{k}/K^{(2)})$. We will show that G_2-coinvariant quotient $(\widetilde{X})_{G_2}$ is pseudo-null. Then by Proposition 2.3, the theorem follows.

We shall recall the argument given in [5]. Let L_2 be the maximal subfield of $L(\widetilde{k})$ which is abelian over $K^{(2)}$. Then $\mathrm{Gal}(L_2/\widetilde{k})$ is isomorphic to $(\widetilde{X})_{G_2}$. For a prime P_2 of $K^{(2)}$ lying above $\mathfrak{P}_2'$, let I_{P_2} be the inertia group of $L_2/K^{(2)}$ for P_2. (I_{P_2} is isomorphic to the additive group of $\mathbb{Z}_p$.) Moreover, we put $\mathcal{I}_2 = \sum_{P_2 \mid \mathfrak{P}_2'} I_{P_2}$. (Note that there are infinitely many primes in $K^{(2)}$ lying above $\mathfrak{P}_2'$.) Since $L_2/K^{(2)}$ is unramified outside the primes lying above $\mathfrak{P}_2'$, we have the following exact sequence of $\mathbb{Z}_p[[\mathrm{Gal}(K^{(2)}/k)]]$-modules:

$$0 \to \mathcal{I}_2 \to \mathrm{Gal}\bigl(L_2/K^{(2)}\bigr) \to \mathrm{Gal}\bigl(L\bigl(K^{(2)}\bigr)/K^{(2)}\bigr) \to 0. \tag{1}$$

Note that $\mathcal{I}_2$ is generated by the inertia groups for the primes lying above $\mathfrak{P}_2'$, and these inertia groups are conjugate under the action of $\mathrm{Gal}(K^{(2)}/k)$. Then $\mathcal{I}_2$ is a finitely generated $\mathbb{Z}_p[[\mathrm{Gal}(K^{(2)}/k)]]$-module. We see that $\mathrm{Gal}(L_2/K^{(2)})$ is also a finitely generated $\mathbb{Z}_p[[\mathrm{Gal}(K^{(2)}/k)]]$-module. (These facts are essentially given in [5]. See the proof of [5, Theorem 1].)

Let D_2 be the decomposition group of $K^{(2)}/k$ for $\mathfrak{P}_2'$. We can see that D_2 is nontrivial and isomorphic to the additive group of $\mathbb{Z}_p$ by using Lemma 2.2. Take a topological generator γ of D_2. Let N be the fixed field of $K^{(2)}$ by D_2, then we can see that there exists a positive integer n_1 such that $N/K_{n_1}^{(1)}$ is a $\mathbb{Z}_p$-extension (that is, $K_{n_1}^{(1)}$ is the decomposition field of $K^{(1)}/k$ for $\mathfrak{P}_2'$).

We note that the $\mathbb{Z}_p[[\mathrm{Gal}(K^{(2)}/k)]]$-modules which appeared in (1) can be seen as modules over $\mathbb{Z}_p[[\mathrm{Gal}(K^{(2)}/K_{n_1}^{(1)})]]$. Here, we will show that $(\widetilde{X})_{G_2}$ is pseudo-null as a $\mathbb{Z}_p[[\mathrm{Gal}(K^{(2)}/K_{n_1}^{(1)})]]$-module. Fix a topological generator σ of $\mathrm{Gal}(K^{(2)}/K^{(1)})$. We can show that $\mathrm{Gal}(K^{(2)}/K_{n_1}^{(1)}) \cong \mathbb{Z}_p^2$ is topologically generated by σ and γ. Take an isomorphism from $\mathbb{Z}_p[[\mathrm{Gal}(K^{(2)}/K_{n_1}^{(1)})]]$ to $\Lambda_2 = \mathbb{Z}_p[[T_1, T_2]]$ which satisfies $\sigma \mapsto 1+T_1$ and $\gamma \mapsto 1+T_2$. We shall identify $\mathbb{Z}_p[[\mathrm{Gal}(K^{(2)}/K_{n_1}^{(1)})]]$ and Λ_2 via this isomorphism. (For a detailed theory of Λ_2-modules, see, e.g., [7,18].)

Since the restriction map $I_{P_2} \to \mathrm{Gal}(\widetilde{k}/K^{(2)})$ is injective, we see $\mathrm{Gal}(K^{(2)}/N)$ acts trivially on I_{P_2} for each prime P_2 lying above $\mathfrak{P}_2'$. This implies that $\gamma - 1$ annihilates $\mathcal{I}_2$. Then $\mathcal{I}_2$ is a finitely generated torsion Λ_2-module. We remark that X_2 is a finitely generated torsion Λ_2-module. Moreover, X_2 is also pseudo-null as a Λ_2-module. This follows from, for example, [13, Lemma 2.3] and an equivalent condition of pseudo-nullity (see the second paragraph of Section 1). (The same fact is also mentioned in [16].) Then by the exact sequence (1), we see that $\mathrm{Gal}(L_2/K^{(2)})$ is a finitely generated torsion Λ_2-module and $\mathcal{I}_2$ is pseudo-isomorphic to $\mathrm{Gal}(L_2/K^{(2)})$. (Note that both of $\mathcal{I}_2$ and $\mathrm{Gal}(L_2/K^{(2)})$ are also finitely generated torsion $\mathbb{Z}_p[\![\mathrm{Gal}(K^{(2)}/k)]\!]$-modules.)

Let L_2' be the subfield of L_2 corresponding to the maximal pseudo-null submodule (as a Λ_2-module) of $\mathrm{Gal}(L_2/K^{(2)})$. Since $\mathcal{I}_2$ is pseudo-isomorphic to $\mathrm{Gal}(L_2/K^{(2)})$, we can see that γ acts trivially on $\mathrm{Gal}(L_2'/K^{(2)})$ by using the same argument given in the proof of Proposition 4.1 (see the fifth paragraph of the proof of Proposition 4.1). Hence we see that L_2' is an abelian extension over N. Note that $\mathrm{Gal}(N/K_{n_1}^{(1)})$ acts on $\mathrm{Gal}(L_2'/K^{(2)})$ because $\mathrm{Gal}(L_2'/K^{(2)})$ is a module over $\mathbb{Z}_p[\![\mathrm{Gal}(K^{(2)}/k)]\!]$.

Lemma 5.1. *L_2'/N is unramified outside the primes lying above $\mathfrak{P}_2'$.*

Proof. Let N' be the unique $\mathbb{Z}_p$-extension of k contained in N. It is sufficient to show that all of $\mathfrak{P}_1$, $\mathfrak{P}_2$, and $\mathfrak{P}_1'$ are ramified in N'/k because all of the inertia groups of $K^{(2)}/k$ for $\mathfrak{P}_1$, $\mathfrak{P}_2$, and $\mathfrak{P}_1'$ are isomorphic to $\mathbb{Z}_p$, and $L_2'/K^{(2)}$ is unramified outside the primes lying above $\mathfrak{P}_2'$.

By Proposition 2.1, there is no $\mathbb{Z}_p$-extension of k in which only one prime of k lying above p ramifies. Then we assume that there are only two primes in k which are ramified in N'/k. However, it is already seen in Lemma 2.2 that every prime of k lying above p is finitely decomposed in N'. It is a contradiction because $\mathfrak{P}_2'$ is decomposed completely in N'. □

Lemma 5.2. $\mathrm{Gal}(L_2'/K^{(2)})$ *is a finitely generated torsion* $\mathbb{Z}_p[\![\mathrm{Gal}(N/K_{n_1}^{(1)})]\!]$*-module.*

Proof. By Lemma 5.1, we see that L_2' is contained in $\mathcal{M}_{\mathfrak{P}_2'}(N)$. We shall show that $\mathrm{Gal}(\mathcal{M}_{\mathfrak{P}_2'}(N)/N)$ is a finitely generated torsion $\mathbb{Z}_p[\![\mathrm{Gal}(N/K_{n_1}^{(1)})]\!]$-module. Then the lemma follows from this claim.

Because $N/K_{n_1}^{(1)}$ is a $\mathbb{Z}_p$-extension, we can identify $\mathbb{Z}_p[\![\mathrm{Gal}(N/K_{n_1}^{(1)})]\!]$ as the one variable power series ring $\mathbb{Z}_p[\![T]\!]$. For a positive integer m, let N_m be the mth layer of $N/K_{n_1}^{(1)}$ and $\mathcal{M}_{\mathfrak{P}_2'}(N_m)$ the maximal abelian pro-p extension of N_m which is unramified outside the primes lying above $\mathfrak{P}_2'$. By using a well-known argument of Iwasawa theory (see, e.g., [9,24]), there exists a positive integer m_0 such that for all $m > m_0$,

$$\mathrm{Gal}\big(\mathcal{M}_{\mathfrak{P}_2'}(N_m)/N_m\big) \cong \mathrm{Gal}\big(\mathcal{M}_{\mathfrak{P}_2'}(N)/N\big)/\nu_{m,m_0}W,$$

where

$$W = \mathrm{Gal}\big(\mathcal{M}_{\mathfrak{P}_2'}(N)/N\mathcal{M}_{\mathfrak{P}_2'}(N_{m_0})\big)$$

and

$$\nu_{m,m_0} = \frac{(1+T)^{p^m}-1}{(1+T)^{p^{m_0}}-1} \in \mathbb{Z}_p[\![T]\!].$$

By Proposition 2.1, we can see that $\mathrm{Gal}(\mathcal{M}_{\mathfrak{P}_2'}(N_m)/N_m)$ is finite for all m. Using the argument given in Chapter 13 of [24] or [9, Lemma 6.1], we can show that $\mathrm{Gal}(\mathcal{M}_{\mathfrak{P}_2'}(N)/N)$ is a finitely generated torsion module over $\mathbb{Z}_p[\![\mathrm{Gal}(N/K_{n_1}^{(1)})]\!]$. □

We shall finish the proof of Theorem 1.1. Recall that σ is a topological generator of $\mathrm{Gal}(K^{(2)}/K^{(1)})$ and γ is a topological generator of $\mathrm{Gal}(K^{(2)}/N)$. Recall also that we fixed an isomorphism

$\mathbb{Z}_p[\![\mathrm{Gal}(K^{(2)}/K^{(1)}_{n_1})]\!]$ to Λ_2 such that $\sigma \mapsto 1 + T_1$ and $\gamma \mapsto 1 + T_2$. Since L'_2/N is an abelian extension, we see that T_2 annihilates $\mathrm{Gal}(L'_2/K^{(2)})$. By Lemma 5.2 and the fact that $\mathrm{Gal}(N/K^{(1)}_{n_1}) \cong \mathrm{Gal}(K^{(2)}/K^{(1)})$, we can take a nonzero element $f(T_1)$ of $\mathbb{Z}_p[\![T_1]\!]$ such that $f(T_1)$ annihilates $\mathrm{Gal}(L'_2/K^{(2)})$. Since $f(T_1)$ and T_2 are relatively prime, $\mathrm{Gal}(L'_2/K^{(2)})$ is annihilated by two relatively prime elements of Λ_2. Hence we showed that $\mathrm{Gal}(L'_2/K^{(2)})$ is a pseudo-null Λ_2-module.

By the definition of L'_2 (see the paragraph before Lemma 5.1), we see that $\mathrm{Gal}(L_2/K^{(2)})$ is a pseudo-null Λ_2-module. Therefore, we conclude that $(\widetilde{X})_{G_2} \cong \mathrm{Gal}(L_2/\widetilde{k})$ is a pseudo-null Λ_2-module. By [13, Lemma 2.3] and an equivalent condition of pseudo-nullity (see the second paragraph of Section 1), we see that $(\widetilde{X})_{G_2}$ is also pseudo-null as a $\mathbb{Z}_p[\![\mathrm{Gal}(K^{(2)}/k)]\!]$-module. Our main theorem has been proved.

Remark. We can prove our main theorem under a weaker condition. That is, if neither $\mathfrak{P}_1$ nor $\mathfrak{P}_2$ splits in $K^{(1)}/k$ and X_1 is finite, then GGC holds for k and p. This follows from Proposition 4.1 and the argument given in Section 5. This result is used in [11].

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